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Bernstein–Durrmeyer Polynomials on a Simplex

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1. INTRODUCTION

Let $S \subset \mathbb{R}^d$ be the simplex defined by

$$S = \{\mathbf{x} \in \mathbb{R}^d \mid x_i \geqslant 0,\ i = 1, 2, ..., d,\ 1 - |\mathbf{x}| \geqslant 0\}.$$

By $L^p(S)$, $1 \leqslant p \leqslant +\infty$, we denote the space of (the equivalence classes of) Lebesgue measurable functions f on S for which the norm $\|f\|_p^p = \int |f|^p$ is finite; $C(S)$ denotes the space of continuous functions on S equipped with the maximum norm. Let $f \in L^1(S)$. For each $n \in \mathbb{N}_o$, the Bernstein–Durrmeyer polynomial of f is defined by

$$\forall \mathbf{x} \in S \qquad V_n(f; \mathbf{x}) = \sum_{|\mathbf{k}| \leqslant n} D_{\mathbf{k}n}(f)\, p_{\mathbf{k}n}(\mathbf{x}), \qquad D_{\mathbf{k}n}(f) = \frac{\int f p_{\mathbf{k}n}}{\int p_{\mathbf{k}n}},$$

where $p_{\mathbf{k}n}(\mathbf{x})$ are the Bernstein basis polynomials in $\mathscr{P}_n(S)$—the subspace of polynomials of degree $\leqslant n$, i.e.,

$$p_{\mathbf{k}n}(\mathbf{x}) = \frac{n!}{\mathbf{k}!(n - |\mathbf{k}|)!} \mathbf{x}^{\mathbf{k}} (1 - |\mathbf{x}|)^{n - |\mathbf{k}|} \qquad \text{and} \qquad \int_S p_{\mathbf{k}n} = \frac{n!}{(n+d)!}.$$

Here and in the following, for $\mathbf{x} = (x_1, x_2, ..., x_d) \in \mathbb{R}^d$ and $\mathbf{k} = (k_1, k_2, ..., k_d) \in \mathbb{N}_o^d$, we denote as usual

$$|\mathbf{x}| = \sum_{i=1}^{d} x_i, \qquad \mathbf{x}^{\mathbf{k}} = x_1^{k_1} x_2^{k_2} \cdots x_d^{k_d},$$

$$|\mathbf{k}| = \sum_{i=1}^{d} k_i, \qquad \mathbf{k}! = k_1!\, k_2! \cdots k_d!.$$

These polynomials were introduced and studied by Derriennic [6] in 1985,

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for the one-dimensional case by Derriennic [5] in 1981. For $d=1$ Ditzian and Ivanov [7] characterized their approximation behavior in 1989. In two papers [2, 3], the first- and the last-named authors considered the Bernstein–Durrmeyer polynomials on $[0, 1]$ with respect to the Jacobi weights and pointed out that the polynomials could be identified with the de la Vallèe–Poussin means of a Jacobi series. A characterization of the approximation behavior was given for the weighted L^p spaces by use of the Peetre K-moduli between the Lebesgue spaces and weighted Sobolev spaces. Going back to the original Bernstein–Durrmeyer polynomials on $[0, 1]$, we have that for each $n \in \mathbb{N}_o$, the Legendre polynomials Q_k, $0 \leqslant k \leqslant n$, are the eigen-functions of V_n, more precisely,

$$V_n Q_k = \lambda_{kn} Q_k, \qquad \text{where} \quad \lambda_{kn} = \frac{n!}{(n-k)!} \frac{(n+1)!}{(n+k+1)!}.$$

From this it follows that for an $f \in L^p(0, 1)$ the approximation behavior of $\{V_n f\}_{n=0}^{\infty}$ can be completely characterized by the Peetre K-modulus $K(f; t; L^p, D^p)$, $t > 0$, where for $1 \leqslant p < +\infty$,

$$D^p(0, 1) = \{f \in L^p(0, 1) \mid f, f' \text{ are l.a.c. on } (0, 1),$$
$$\varphi^2(x) f'(x) \to 0 \text{ as } x \to 0+, 1-, \text{ and } Uf \in L^p(0, 1)\},$$

while for $p = \infty$,

$$D[0, 1] = \{f \in C[0, 1] \mid f \in C^2(0, 1),$$
$$\varphi^2(x) f'(x) \to 0 \text{ as } x \to 0+, 1-, \text{ and } Uf \in C[0, 1]\}.$$

Here, U is the differential operator associated with the Legendre polynomials; i.e., $Uf = (\varphi^2 f')'$, $\varphi(x) = \sqrt{x(1-x)}$, and $D^p(0, 1)$ and $D[0, 1]$ are equipped with the semi-norms $\|Uf\|_p$ and $\|Uf\|_\infty$, respectively.

It seems commonly more satisfactory to have a characterization of the approximation behavior via moduli of smoothness. The main effort in [2, 3] was thus devoted to replacing $K(f; t; L^p, D^p)$ by an appropriate modulus of smoothness, as defined by Ditzian and Totik [8, Chap. 2]. The latter is known to be equivalent to the K-modulus $K(f; t; L^p, W_\varphi^{p,2})$, where for $1 \leqslant p < +\infty$,

$$W_\varphi^{p,2}(0, 1) = \{f \in L^p(0, 1) \mid f, f' \text{ are l.a.c. on } (0. 1), \text{ and } \varphi^2 f'' \in L^p(0, 1)\},$$

while for $p = +\infty$,

$$C_\varphi^2[0, 1] = \{f \in C[0, 1] \mid f \in C^2(0, 1), \varphi^2(x) f''(x) \to 0 \text{ as } x \to 0+, 1-\}.$$

In the present paper, we follow this outline to study $\{V_n\}_{n=0}^{\infty}$ on the simplex $S \subset \mathbb{R}^d$, $d \geqslant 2$. It turns out that the abstract theory is almost independent of the dimension; i.e., for an $f \in L^p(S)$ the approximation behavior of $\{V_n f\}_{n=0}^{\infty}$ can be characterized completely by a d-dimensional analogue of $K(f; t; L^p, D^p)$, $t > 0$, where $D^p(S)$ is given abstractly via the Legendre coefficients of f on S. A characterization of $D^p(S)$ by means of differentiability properties seems to be more difficult as compared to $D^p(0, 1)$; a characterization of the approximation behavior of $\{V_n f\}_{n=0}^{\infty}$ through a modulus of smoothness presents further difficulties. We shall give a complete answer only for $f \in L^2(S)$.

2. Simple Properties and Saturation

First we list some basic properties of the operators $\{V_n\}_{n=0}^{\infty}$, cf. [6].

(1) $\forall n \in \mathbb{N}_o$, $V_n : L^p(S) \to \mathscr{P}_n$, $1 \leqslant p \leqslant +\infty$, is a positive, linear contraction. In particular,

$$V_n(1, \mathbf{x}) = 1 \qquad \text{and} \qquad V_n(x_j, \mathbf{x}) = \frac{1 + n x_j}{n + d + 1}, \qquad 1 \leqslant j \leqslant d.$$

It follows that the sequence $\{V_n\}_{n=0}^{\infty}$ forms an approximate identity on $L^p(S)$, $1 \leqslant p < \infty$, and $C(S)$, $p = \infty$.

(2) $\forall k, n \in \mathbb{N}_o$, $V_n(\mathscr{P}_k) \subset \mathscr{P}_k$. Moreover, for $\mathbf{m} \in \mathbb{N}_o^d$, $|\mathbf{m}| \leqslant n$,

$$V_n(\mathbf{x}^{\mathbf{m}}; \mathbf{x}) = \frac{(n+d)!}{(n+d+|\mathbf{m}|)!} \sum_{\mathbf{s}=0}^{\mathbf{m}} \frac{\mathbf{m}!}{\mathbf{s}!} \binom{\mathbf{m}}{\mathbf{s}} \frac{n!}{(n - |\mathbf{s}|)!} \mathbf{x}^{\mathbf{s}},$$

where

$$\mathbf{s} \in \mathbb{N}_o^d, \ \sum_{s=0}^{\mathbf{m}} = \sum_{s_1=0}^{m_1} \cdots \sum_{s_d=0}^{m_d} \qquad \text{and} \qquad \binom{\mathbf{m}}{\mathbf{s}} = \frac{\mathbf{m}!}{\mathbf{s}!(\mathbf{m}-\mathbf{s})!}.$$

(3) With respect to the bilinear form $\langle f, g \rangle = \int fg$, $\forall n \in \mathbb{N}_o$, the operator V_n is self-adjoint, i.e.,

$$\forall f, g \in L^1(S), \qquad \langle V_n f, g \rangle = \langle f, V_n g \rangle.$$

For every $k \geqslant 1$, we denote by $\mathscr{E}_k$ the subspace of $\mathscr{P}_k$ which is orthogonal to $\mathscr{P}_{k-1}$ on S w.r.t. the inner product $\langle \cdot, \cdot \rangle$. Furthermore, for each $k \in \mathbb{N}_o$, $\mathbf{m} \in \mathbb{N}_o^d$, and $|\mathbf{m}| = k$, we denote the polynomials in $\mathscr{E}_k$ with a monic leading term by $Q_{\mathbf{m}}^{(k)}$, i.e.,

$$Q_{\mathbf{m}}^{(k)}(\mathbf{x}) = \mathbf{x}^{\mathbf{m}} + q(\mathbf{x}) \in \mathscr{E}_k, \qquad \text{where} \quad q \in \mathscr{P}_{k-1}.$$

We set

$$h^{(k)}_{\mathbf{m},\mathbf{m}'} = \langle Q^{(k)}_{\mathbf{m}}, Q^{(k)}_{\mathbf{m}'} \rangle, \qquad |\mathbf{m}| = |\mathbf{m}'| = k.$$

For a function $f \in L^1(S)$, its series expansion w.r.t. $\{Q^{(k)}_{\mathbf{m}} \mid \mathbf{m} \in \mathbb{N}^d_o, |\mathbf{m}| = k\}$ is defined by

$$\sum_{k=0}^{\infty} \sum_{|\mathbf{m}|=k} a^{(k)}_{\mathbf{m}}(f)\, Q^{(k)}_{\mathbf{m}},$$

where the functionals $a^{(k)}_{\mathbf{m}}$ are determined by the linear system of equations

$$\sum_{|\mathbf{m}'|=k} h^{(k)}_{\mathbf{m},\mathbf{m}'} a^{(k)}_{\mathbf{m}'}(f) = \langle f, Q^{(k)}_{\mathbf{m}} \rangle =: A^{(k)}_{\mathbf{m}}(f), \qquad |\mathbf{m}| = k.$$

The coefficient matrix of this system

$$H^{(k)} = (h^{(k)}_{\mathbf{m},\mathbf{m}'})_{|\mathbf{m}|=k, |\mathbf{m}'|=k}$$

is of the size $e_k \times e_k$, where $e_k = \binom{k+d-1}{d-1}$ is the number of indices $\mathbf{m} \in \mathbb{N}^d_o$ such that $|\mathbf{m}| = k$. Since $h^{(k)}_{\mathbf{m},\mathbf{m}'} = h^{(k)}_{\mathbf{m}',\mathbf{m}}$, an order can be given to $\{\mathbf{m} \in \mathbb{N}^d_o \mid |\mathbf{m}| = k\}$, e.g., the lexicographical order, such that the matrix $H^{(k)}$ is symmetric. Once this order is given, the matrix is clearly positive definite and $\{a^{(k)}_{\mathbf{m}} \mid \mathbf{m} \in \mathbb{N}^d_o, |\mathbf{m}| = k\}$ are continuous linear functional on $L^1(S)$. It now follows from (2) and (3) (cf. [6]) that

(4) $\forall k \in \mathbb{N}_o$, the space $\mathscr{E}_k$ is an eigen-space of V_n associated to the eigen-value

$$\lambda_{k,n} = \frac{n!}{(n-k)!} \frac{(n+d)!}{(n+k+d)!}, \qquad 0 \leqslant k \leqslant n;$$

i.e., $\forall \mathbf{m} \in \mathbb{N}^d_o$, $|\mathbf{m}| = k$, the polynomial $Q^{(k)}_{\mathbf{m}}$ is an eigen-function of $\lambda_{k,n}$. In particular,

$$V_n(f; \mathbf{x}) = \sum_{k=0}^{n} \lambda_{kn} \sum_{|\mathbf{m}|=k} a^{(k)}_{\mathbf{m}}(f)\, Q^{(k)}_{\mathbf{m}}(\mathbf{x}).$$

It follows from Appell and Kampé de Fériet [1, Chaps. VI, VII] that $Q^{(k)}_{\mathbf{m}}$ satisfies the partial differential equation

$$UQ^{(k)}_{\mathbf{m}} = -k(k+d)\, Q^{(k)}_{\mathbf{m}}, \qquad \mathbf{m} \in \mathbb{N}^d_o, |\mathbf{m}| = k,$$

where

$$U = \sum_{1 \leqslant i \leqslant d} x_i(1-x_i)\, D_i^2 - 2 \sum_{1 \leqslant i < j \leqslant d} x_i x_j D_i D_j + \sum_{1 \leqslant i \leqslant d} (1-(d+1)\, x_i)\, D_i$$

and $D_i = \partial/\partial x_i$. Therefore,

(5) $UV_n f = V_n Uf$, and

(6) $\forall k \in \mathbb{N}_o$,

$$\lim_{n \to \infty} n\{\lambda_{k,n} - 1\} = -k(k+d);$$

i.e., the Bernstein–Durrmeyer polynomials show a saturation behavior.

In their book [1] on hypergeometrical and hyperspherical functions from 1926, Appell and K. de Fériet defined and studied multivariate Jacobi polynomials over the simplex S with respect to the weight function $w(\mathbf{x}) = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d}(1 - |\mathbf{x}|)^{\beta}$, $\alpha_j, \beta > -1$, $1 \leqslant j \leqslant d$. We considered here $\alpha_j = \beta = 0$, $1 \leqslant j \leqslant d$, which corresponds to multivariable Legendre polynomials. For this reason, we call $A_{\mathbf{m}}^{(k)}(f)$ the Legendre coefficients of f, and $V_n f$ can be viewed as the de la Vallée–Poussin means of the Legendre series of f on S. We take $\{Q_{\mathbf{m},k} \mid \mathbf{m} \in \mathbb{N}_o^d, |\mathbf{m}| = k\}$ as a basis for $\mathscr{E}_k$, for a purpose which will become clear in Section 3.

Appell and K. de Fériet gave another interesting basis for $\mathscr{E}_k$, namely,

$$F_{\mathbf{m}}^{(k)}(\mathbf{x}) = D^{\mathbf{m}}(\mathbf{x}^{\mathbf{m}}(1 - |\mathbf{x}|)^{\mathbf{m}}), \qquad \mathbf{m} \in \mathbb{N}_o^d, |\mathbf{m}| = k,$$

where $D^{\mathbf{m}} = \partial^{|\mathbf{m}|}/\partial x_1^{m_1} \cdots \partial x_d^{m_d}$, which is the multivariable analogue of Rodrigues' formula for the Legendre polynomials. In addition, $\{Q_{\mathbf{m}}^{(k)} \mid \mathbf{m} \in \mathbb{N}_o^d, |\mathbf{m}| = k\}$ and $\{F_{\mathbf{m}}^{(k)}, \mathbf{m} \in \mathbb{N}_o^d, |\mathbf{m}| = k\}$, $k \in \mathbb{N}_o$, form a biorthogonal system on S, i.e.,

$$\langle Q_{\mathbf{m}}^{(k)}, F_{\mathbf{m}'}^{(k')} \rangle = 0, \qquad k \neq k' \text{ or } \mathbf{m} \neq \mathbf{m}'.$$

Appell and K. de Fériet gave the definition of $F_{\mathbf{m}}^{(k)}$ and of its differential equation explicitly only for $d = 2$. The case $d > 2$ follows easily from their Chapter VII on hypergeometric series.

The double-sequence of multipliers has the property

$$\forall k \in \mathbb{N}_o \text{ and } \forall n \in \mathbb{N}, \qquad \lambda_{k,n} - \lambda_{k,n-1} = \frac{k(k+d)}{n(n+d)} \lambda_{kn},$$

and consequently, $\forall f \in L^1(S)$ and $\forall n \in \mathbb{N}_o$

$$(7) \quad V_{n-1} f - V_n f = \frac{UV_n f}{n(n+d)}, \qquad \text{or} \qquad V_n f - f = \sum_{k=n+1}^{\infty} \frac{UV_k f}{k(k+d)}.$$

We now define the subspaces $D^p(S)$ of $L^p(S)$, $1 \leqslant p \leqslant +\infty$, and $D(S)$ of $C(S)$, respectively, through

$$D^p(S) = \{f \in L^p(S) \mid \exists g \in L^p(S)$$
$$\ni \forall k \in \mathbb{N}_o, \forall \mathbf{m} \in \mathbb{N}_o^d, |\mathbf{m}| = k, -k(k+d)\, A_{\mathbf{m}}^{(k)}(f) = A_{\mathbf{m}}^{(k)}(g)\},$$

and the operator $\mathscr{U}$ from $L^p(S)$ into itself through $\mathscr{U}f = g$; similarly on $C(S)$. $\mathscr{U}$ is a closed linear operator with dense domain $D^p(S)$. Indeed, $\mathscr{P} \subset D^p(S)$ and $\forall p \in \mathscr{P}$, $\mathscr{U}p = Up$; i.e., $\mathscr{U}$ is the closed linear extension of U.

THEOREM 1. (a) *For* $f \in D^p(S)$, $1 \leqslant p < +\infty$, *and* $f \in D(S)$, $p = \infty$, *respectively,*

$$\lim_{n \to \infty} (n+1)\{V_n f - f\} = \mathscr{U}f \qquad \textit{in norm.}$$

(b) *Let* $f \in L^1(S)$ *be such that*

$$\|V_n f - f\|_1 = o\left(\frac{1}{n+1}\right) \qquad (n \to \infty),$$

then $f(\mathbf{x}) = \text{const}$ *a.e.*

(c) *Let* $f \in L^p(S)$, $1 \leqslant p < \infty$, *and* $f \in C(S)$, $p = \infty$, *respectively. The function* f *belongs to the saturation class, i.e.,*

$$\|V_n f - f\|_p = O\left(\frac{1}{n+1}\right) \qquad (n \to \infty),$$

exactly when $f \in D^p(S)$ *for* $1 < p \leqslant +\infty$, *while for* $p = 1$, f *belongs to*

$$\{f \in L^1(S) \mid \exists \mu \in BV(S) \ni \forall k \in \mathbb{N}_o, \mathbf{m} \in \mathbb{N}_o^d, |\mathbf{m}| = k, \\ -k(k+d)\, A_{\mathbf{m}}^{(k)}(f) = A_{\mathbf{m}}^{(k)}(d\mu)\}.$$

Furthermore, the approximation behavior of $\{V_n\}_{n=0}^{\infty}$ on $L_p(S)$, $1 \leqslant p < +\infty$, and $C(S)$, respectively, is completely described by the Peetre K-modulus

$$K(f; t; L^p, D^p) = \inf_{g \in D^p} \{\|f - g\|_p + t\, \|\mathscr{U}g\|_p\}, \qquad t > 0.$$

More precisely,

THEOREM 2. *Let* $f \in L^p(S)$, $1 \leqslant p < +\infty$, *and* $f \in C(S)$, $p = +\infty$, *respectively.*

$$\forall n \in \mathbb{N}_o, \qquad \|V_n f - f\|_p \leqslant \text{const}\, K\left(f; \frac{1}{n+1}; L^p, D^p\right),$$

and conversely,

$$K\left(f; \frac{1}{n+1}; L^p, D^p\right) \leqslant \text{const} \max_{n \leqslant k} \{\|V_k f - f\|_p\},$$

where the constants depend only on p.

These theorems[1] are proved in almost complete analogy to the one-variable theory given in [2].

3. Characterization of $D^2(S)$

To simplify the problem, we shall state and prove our results for $d=2$ only. For $\mathbf{x} \in \mathbb{R}^2$, we write $\mathbf{x}=(x, y)$, for $Q_{\mathbf{m}}^{(k)}(\mathbf{x}), \mathbf{m} \in \mathbb{N}_o^2$, and $|\mathbf{m}|=k, k \in \mathbb{N}_o$,

$$Q_j^{(k)}(x, y)=x^j y^{k-j}+\cdots, \qquad 0 \leqslant j \leqslant k,$$

and correspondingly, $a_j^{(k)}$, $A_j^{(k)}$, and $H^{(k)}=(h_{ij}^{(k)})_{i,j=0}^k$, where $h_{ij}^{(k)}=\langle Q_i^{(k)}, Q_j^{(k)}\rangle$. Furthermore, denoting by $a^{(k)}$ and $A^{(k)}$ vectors with coefficients given by the linear functionals $a_j^{(k)}$ and $A_j^{(k)}$, $0 \leqslant j \leqslant k$, respectively, we have

$$H^{(k)} a^{(k)}=A^{(k)}.$$

The differential operator U can be written as

$$U=\frac{\partial}{\partial x}\left\{x(1-x) \frac{\partial}{\partial x}-x y \frac{\partial}{\partial y}\right\}+\frac{\partial}{\partial y}\left\{y(1-y) \frac{\partial}{\partial y}-x y \frac{\partial}{\partial x}\right\}.$$

For our purposes, however, a more useful form is

$$\begin{aligned} U &= D_1[x(1-x-y) D_1]+D_2[y(1-x-y) D_2]+D_3[x y D_3] \\ &=: U_1+U_2+U_3, \end{aligned}$$

where $D_1=\partial/\partial x$, $D_2=\partial/\partial y$, and $D_3=\partial/\partial x-\partial/\partial y$. We have

Lemma 1. *The differential operators U_i, $i=1,2,3$, are self-adjoint and commute with V_n, $n \in \mathbb{N}_o$.*

The statements of the lemma can be easily checked. Next we have

Lemma 2. *Let $Q_{-1}^{(k)}=Q_{k+1}^{(k)}:=0$. Then for $0 \leqslant i \leqslant k$, $k \in \mathbb{N}_o$,*

$$\begin{aligned} U_1 Q_i^{(k)} &= -i^2 Q_{i-1}^{(k)}-i(i+1) Q_i^{(k)}, \\ U_2 Q_i^{(k)} &= -(k-i)^2 Q_{i+1}^{(k)}-(k-i)(k-i+1) Q_i^{(k)}, \end{aligned}$$

[1] While completing the paper we learned that the theorems have been formulated and proved independently by W. Chen and Z. Ditzian as well as by M. M. Derriennic. These authors also study linear combinations of the Bernstein–Durrmeyer operator to achieve higher order of approximation, while our emphasis lies on the characterization of the domain of definition of the operator $\mathscr{U}$.

and

$$U_3 Q_i^{(k)} = i^2 Q_{i-1}^{(k)} + (k-i)^2 Q_{i+1}^{(k)} - [k + 2i(k-i)]\, Q_i^{(k)}.$$

Proof. We prove only the first equality. Clearly, $U_1 : \mathscr{P}_k \to \mathscr{P}_k$, $U_1 Q_0^{(k)} \in \mathscr{P}_{k-1}$, and for $0 < i \leqslant k$

$$U_1 Q_i^{(k)} = -i(i+1)\, Q_i^{(k)} - i^2 Q_{i-1}^{(k)} + \text{lower terms},$$

where "lower terms" means a polynomial in $\mathscr{P}_{k-1}$. Thus, by the self-adjointness of U_1 and the orthogonality of the eigen-spaces $\mathscr{E}_k$, we get for all $0 \leqslant j \leqslant k$, $k \in \mathbb{N}_o$

$$A_j^{(m)}(U_1 Q_i^{(k)}) = 0, \qquad m \neq k,$$
$$A_j^{(k)}(U_1 Q_i^{(k)}) = -i(i+1)\, h_{j,i}^{(k)} - i^2 h_{j,i-1}^{(k)}, \qquad \text{where} \quad h_{j,-1}^{(k)} := 0.$$

It follows that for the function $U_1 Q_i^{(k)}$

$$a_j^{(m)}(U_1 Q_i^{(k)}) = 0,\ m \neq k, \qquad a_j^{(k)}(U_1 Q_i^{(k)}) = 0,\ j \neq i, i-1,$$

while

$$a_i^{(k)}(U_1 Q_i^{(k)}) = -i(i+1) \qquad \text{and} \qquad a_{i-1}^{(k)}(U_1 Q_i^{(k)}) = -i^2. \quad \blacksquare$$

The splitting of the operator $\mathscr{U}$ into the three parts $\mathscr{U}_1$, $\mathscr{U}_2$, and $\mathscr{U}_3$ allows us to reduce the two-dimensional problem to one-dimensional problems. We therefore define for $1 \leqslant p < \infty$, the following subspaces of $L^p(S)$,

$$D_1^p(S) = \{ f \in L^p(S) \mid \exists g_1 \in L^p(S)$$
$$\ni \forall k \in \mathbb{N}_o, 0 \leqslant j \leqslant k,\ A_j^{(k)}(g_1) = -j^2 A_{j-1}^{(k)}(f) - j(j+1)\, A_j^{(k)}(f) \},$$
$$D_2^p(S) = \{ f \in L^p(S) \mid \exists g_2 \in L^p(S)$$
$$\ni \forall k \in \mathbb{N}_o, 0 \leqslant j \leqslant k,\ A_j^{(k)}(g_2) = -(k-j)^2 A_{j+1}^{(k)}(f) - (k-j)(k-j+1)\, A_j^{(k)}(f) \},$$

and

$$D_3^p(S) = \{ f \in L^p(S) \mid \exists g_3 \in L^p(S)$$
$$\ni \forall k \in \mathbb{N}_o, 0 \leqslant j \leqslant k,\ A_j^{(k)}(g_3) = j^2 A_{j-1}^{(k)}(f) + (k-j)^2 A_{j+1}^{(k)}(f) - [k + 2j(k-j)]\, A_j^{(k)}(f) \}.$$

Furthermore, we define the linear operators $\mathscr{U}_i$ with domain $D_i^p(S)$ through

$\mathcal{U}_i f = g_i$, $i = 1, 2, 3$, and similarly, for $p = \infty$, the subspaces $D_i(S)$ of $C(S)$ and the operators $\mathcal{U}_i$, $i = 1, 2, 3$.

It follows from Lemma 2 and Fubini's theorem that

$$\begin{aligned} D_1^p(S) = \{ f \in L^p(S) \mid D_1 f,\ D_1^2 f \in L_{\text{loc}}(\mathring{S}), \\ \text{for almost all } 0 < y < 1,\ x(1-y-x)\, D_1 f(x, y) \to 0 \\ \text{as } x \to 0+,\ (1-y)-, \text{ and } \mathcal{U}_1 f \in L^p(S) \}, \end{aligned}$$

where the derivatives are in the distributional sense. Analogous characterizations hold for $D_2^p(S)$ and $D_3^p(S)$.

Since $\mathcal{U} = \mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_3$, we have

$$\bigcap_{i=1}^{3} D_i^p(S) \subset D^p(S) \qquad \text{and} \qquad \bigcap_{i=1}^{3} D_i(S) \subset D(S).$$

For $p = 2$ we even have equality.

LEMMA 3.

$$\bigcap_{i=1}^{3} D_i^2(S) = D^2(S).$$

Proof. We show $D_1^2(S) \supset D^2(S)$. Let $f \in D^2(S)$. We set $g = \mathcal{U} f$. By orthogonality,

$$\| f \|_2^2 = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \sum_{i=0}^{k} a_j^{(k)}(f)\, a_i^{(k)}(f)\, h_{ij}^{(k)} = \sum_{k=0}^{\infty} X_k^T H^{(k)} X_k,$$

where $X_k = a^{(k)}(f)$, and

$$\| g \|_2^2 = \sum_{k=1}^{\infty} k^2 (k+2)^2\, X_k^T H^{(k)} X_k.$$

Let g_1 be the function defined through

$$A_j^{(k)}(g_1) = -j^2 A_{j-1}^{(k)}(f) - j(j+1)\, A_j^{(k)}(f), \qquad 0 \leqslant j \leqslant k,\ k \in \mathbb{N}_o.$$

We claim that $g_1 \in L^2(S)$. To prove the claim, we write

$$A^{(k)}(g_1) = B_k A^{(k)}(f) \qquad \forall k \in \mathbb{N}_o,$$

where B_k is a lower triangular matrix in $\mathbb{R}^{(k+1) \times (k+1)}$ with main diagonal given by $(0, -2, -6, \ldots, -k(k+1))$. Note that the matrix B_k satisfies

$$A^{(k)}(U_1 Q_i^{(k)}) = B_k A^{(k)}(Q_i^{(k)}), \qquad 0 \leqslant i \leqslant k,$$

and taking into account that $A_j^{(k)}(U_1 Q_i^{(k)}) = A_i^{(k)}(U_1 Q_j^{(k)})$, $0 \leqslant i, j \leqslant k$, $k \in \mathbb{N}_o$, it follows that

$$B_k H^{(k)} = H^{(k)} B_k^T.$$

Therefore, for $Y_k := a^{(k)}(g_1)$

$$H^{(k)} Y_k = A^{(k)}(g_1) = B_k A^{(k)}(f) = B_k H^{(k)} a^{(k)}(f) = H^{(k)} B_k^T X_k,$$

giving $Y_k = B_k^T X_k$. We now prove that for each $k \in \mathbb{N}_o$,

$$Y_k^T H^{(k)} Y_k \leqslant k^2(k+2)^2\, X_k^T H^{(k)} X_k,$$

from which our claim follows. To do so, we shall suppress the sub- and superscript k for the rest of the proof. Since both BHB^T and H are symmetric and since H is positive definite, there exists an invertible matrix S such that

$$H = S^T S \qquad \text{and} \qquad BHB^T = S^T \Lambda S,$$

where $\Lambda = \operatorname{diag}(\lambda_0, \lambda_1, ..., \lambda_k)$. Since, furthermore, $BH = HB^T = S^T SB^T$, we have

$$\Lambda = (S^T)^{-1}\, BHB^T S^{-1} = S(B^T)^2\, S^{-1}.$$

Therefore it follows from the particular form of B that

$$\|\Lambda\| := \max_{0 \leqslant i \leqslant k} |\lambda_i| = k^2(k+1)^2.$$

Hence,

$$\begin{aligned} Y^T H Y = X^T BHB^T X &= (SX)^T \Lambda (SX) \\ &\leqslant \|\Lambda\|\, (SX)^T SX = k^2(k+1)^2\, X^T H X, \end{aligned}$$

or $g_1 \in L^2(S)$ and $\|g_1\|_2 \leqslant \|g\|_2$. Similarly, we obtain that the functions g_2 and g_3 are in $L^2(S)$ and that $\|g_2\|_2, \|g_3\|_2 \leqslant \|g\|_2$. ∎

In other words,

COROLLARY. *For each $f \in D^2(S)$, $f \in D_i^2(S)$, $1 \leqslant i \leqslant 3$, and*

$$\|\mathscr{U}_i f\|_2 \leqslant \|\mathscr{U} f\|_2 \leqslant \sum_{i=1}^{3} \|\mathscr{U}_i f\|_2.$$

We can now characterize the saturation class of the Bernstein–

Durrmeyer polynomials in $L^2(S)$ by means of differentiability properties of the functions. Indeed, if we set

$$\varphi_1(x, y)=\sqrt{x(1-y-x)}, \qquad \varphi_2(x, y)=\sqrt{y(1-y-x)},$$
$$\varphi_3(x, y)=\sqrt{xy}, \qquad (x, y)\in S,$$

it follows easily from the definitions of the subspaces $D_i^2(S)$, $i=1, 2, 3$, above and from Lemmas 4 and 5 of [2] (see the following section):

THEOREM 3. *The saturation class $D^2(S)$ of the Bernstein–Durrmeyer polynomials in $L^2(S)$ is equivalent to the Sobolev space*

$$W_{\Phi}^{2,2}(S)=\{f\in L^2(S) : D_1 f, D_2 f, D_1^2 f, D_2^2 f, D_1 D_2 f\in L_{\text{loc}}(\mathring{S}),$$
$$\text{and } \varphi_1^2 D_1^2 f, \varphi_2^2 D_2^2 f, \varphi_3^2 D_3^2 f\in L^2(S)\}.$$

Moreover, there exists a constant such that $\forall f\in D^2(S)$

$$\frac{1}{\text{const}}\|\varphi_i^2 D_i^2 f\|_2 \leqslant \|\mathscr{U} f\|_2 \leqslant \text{const}\left[\|f\|_2+\sum_1^3 \|\varphi_i^2 D_i^2 f\|_2\right].$$

4. CHARACTERIZATION THROUGH MODULI OF SMOOTHNESS

For $\mathbf{x}\in S\subset\mathbb{R}^d$ we write

$$\varphi_i(\mathbf{x})=\varphi_{ii}(\mathbf{x}) := \sqrt{x_i(1-|\mathbf{x}|)}, \qquad 1\leqslant i\leqslant d,$$

and

$$\varphi_{ij}(\mathbf{x}) := \sqrt{x_i x_j}, \qquad 1\leqslant i<j\leqslant d,$$

$D_i=D_{ii} := \partial/\partial x_i$, $1\leqslant i\leqslant d$, and $D_{ij}=D_i-D_j$, $1\leqslant i<j\leqslant d$. Furthermore, the operator U can be rewritten as

$$U=\sum_{1\leqslant i\leqslant d} D_i(x_i(1-|\mathbf{x}|)\, D_i)+\sum_{1\leqslant i<j\leqslant d} D_{ij}(x_i x_j D_{ij})$$
$$=: \sum_{1\leqslant i\leqslant d} U_i+\sum_{1\leqslant i<j\leqslant d} U_{ij}.$$

For a function $f\in D^p(S)$, the smoothness of Uf is determined by the second order terms of differentiation, i.e., by

$$\sum_{1\leqslant i\leqslant d} x_i(1-|\mathbf{x}|)\, D_i^2 f+\sum_{1\leqslant i<j\leqslant d} x_i x_j D_{ij}^2 f=\sum_{1\leqslant i\leqslant j\leqslant d} \varphi_{ij}^2 D_{ij}^2 f.$$

We thus define for $1 \leqslant p \leqslant \infty$ the Sobolev spaces

$$W_\Phi^{p,2} = \{ f \in L^p(S) \mid D_i f,\ D_{ij} f,\ D_i^2 f,\ D_{ij}^2 f \in L_{\mathrm{loc}}(S),$$
$$\text{and } \varphi_{ij}^2 D_{ij}^2 f \in L^p(S),\ 1 \leqslant i \leqslant j \leqslant d\}.$$

First we prove

LEMMA 4. *Let $1 \leqslant p < +\infty$. $W_\Phi^{p,2} \subset D^p(S)$, and*

$$\forall f \in W_\Phi^{p,2}, \qquad \|Uf\|_p \leqslant \mathrm{const}\left[\|f\|_p + \sum_{1 \leqslant i \leqslant j \leqslant d} \|\varphi_{ij}^2 D_{ij}^2 f\|_p\right].$$

Proof. By the factorization of Uf given above, we have

$$\|Uf\|_p \leqslant \sum_{1 \leqslant i \leqslant d} \|U_i f\|_p + \sum_{1 \leqslant i < j \leqslant d} \|U_{ij} f\|_p.$$

For the proof we denote the differential operator U for the one-dimensional case by U^*; i.e., we write

$$U^* g = [\varphi^2 g']', \qquad \text{where} \quad \varphi(z) = \sqrt{z(1-z)},\ 0 < z < 1.$$

First, we estimate the term $U_1 f$. For $\mathbf{x} = (x_1, x_2, ..., x_d) \in \mathbb{R}^d$, we write $\mathbf{x}_1 = (x_2, ..., x_d)$ and denote the associated face S_1 by

$$S_1 = \{\mathbf{x}_1 : \mathbf{x} = (x_1, \mathbf{x}_1) \in S\}.$$

Changing the variable x_1 into z by setting $x_1 = (1 - |\mathbf{x}_1|)\, z$, $0 < z < 1$, for fixed $\mathbf{x}_1$, we have

$$\begin{aligned}
\|U_1 f\|_p^p &= \int_S |U_1 f(\mathbf{x})|^p \, d\mathbf{x} \\
&= \int_{S_1} d\mathbf{x}_1 \int_0^{1-|\mathbf{x}_1|} |D_1[x_1(1-|\mathbf{x}|)\, D_1 f(\mathbf{x})]|^p \, dx_1 \\
&= \int_{S_1} (1 - |\mathbf{x}_1|)\, d\mathbf{x}_1 \int_0^1 |U^* g(z)|^p \, dz,
\end{aligned}$$

where $g(z) = f((1 - |\mathbf{x}_1|)\, z, x_2, ..., x_d)$. We now apply Lemma 3 of [2] to the last integral, and get

$$\begin{aligned}
\|U_1 f\|_p^p &\leqslant \mathrm{const} \int_{S_1} (1 - |\mathbf{x}_1|)\, d\mathbf{x}_1 \left[\int_0^1 |g(z)|^p \, dz + \int_0^1 |\varphi^2(z)\, g''(z)|^p \, dz\right] \\
&= \mathrm{const} \int_{S_1} d\mathbf{x}_1 \left[\int_0^{1-|\mathbf{x}_1|} |f(\mathbf{x})|^p \, dx_1 + \int_0^{1-|\mathbf{x}_1|} |\varphi_1^2(\mathbf{x})\, D_1^2 f(\mathbf{x})|^p \, dx_1\right] \\
&= \mathrm{const}[\|f\|_p^p + \|\varphi_1^2 D_1^2 f\|_p^p].
\end{aligned}$$

By symmetry, this inequality can be established for all $U_i f$, $1 \leqslant i \leqslant d$, and also for all $U_{ij} f$, $1 \leqslant i < j \leqslant d$, by noting that under the linear transformation $T_i: x_i \to 1 - |\mathbf{x}|$ and $x_j \to x_j$, $j \neq i$,

$$U_{ji} f = U_j(f \circ T_i). \quad \blacksquare$$

Similarly, by Lemma 4 of [2], we have

LEMMA 5. *Let* $1 < p \leqslant +\infty$. *For a function* $f \in L^p(S)$

$$\|\varphi_{ij}^2 D_{ij}^2 f\|_p \leqslant \text{const} \|U_{ij} f\|_p, \qquad 1 \leqslant i \leqslant j \leqslant d,$$

whenever the left-hand side is defined.

With these results, Theorem 3 can be written for arbitrary dimensions as

THEOREM 3bis. $D^2(S) = W_\Phi^{2,2}(S)$, *and for each* $f \in W_\Phi^{2,2}(S)$,

$$\frac{1}{\text{const}} \sum_{1 \leqslant i \leqslant j \leqslant d} \|\varphi_{ij}^2 D_{ij}^2 f\|_2 \leqslant \|Uf\|_2 \leqslant \text{const} \left[\|f\|_2 + \sum_{1 \leqslant i \leqslant j \leqslant d} \|\varphi_{ij}^2 D_{ij}^2 f\|_2 \right].$$

In particular, the approximation behavior of $\{V_n\}_{n=0}^\infty$ *on* $L^2(S)$ *is completely characterized by the K-modulus* $K(f; 1/(n+1); L^2, W_\Phi^{2,2})$.

Let $f \in L^1(S)$. We define for almost all $\mathbf{x} \in S$,

$$\Delta_{t\mathbf{e}}^2 f(\mathbf{x}) = \begin{cases} f(\mathbf{x} + t\mathbf{e}) - 2f(\mathbf{x}) + f(\mathbf{x} - t\mathbf{e}), & x \pm t\mathbf{e} \in S, \\ 0, & \text{otherwise}, \end{cases}$$

where $\mathbf{e}$ is a fixed vector in $\mathbb{R}^d$ and $0 < t < 1$. Furthermore, we define the modulus of smoothness with weighted increments by (cf. Chap. XII in [8])

$$\omega_{\varphi_{ij}\mathbf{e}_{ij}}^2(f; \delta)_p = \sup_{0 < t \leqslant \delta} \|\Delta_{t\varphi_{ij}\mathbf{e}_{ij}}^2 f\|_p, \qquad 1 \leqslant i \leqslant j \leqslant d,$$

and

$$\omega_\Phi^2(f; \delta)_p = \sum_{1 \leqslant i \leqslant j \leqslant d} \omega_{\varphi_{ij}\mathbf{e}_{ij}}^2(f; \delta)_p, \qquad 1 \leqslant p \leqslant +\infty,$$

where $\mathbf{e}_{ii} = \mathbf{e}_i$ and $\mathbf{e}_{ij} = \mathbf{e}_j - \mathbf{e}_i$, $i < j$, with $\mathbf{e}_i$ the ith natural basis vector in $\mathbb{R}^d$, $1 \leqslant i \leqslant d$. We have

LEMMA 6. *Let $1 \leqslant p \leqslant +\infty$. There exists a constant, only dependent on p, such that $\forall f \in L^p$, and $\delta > 0$,*

$$\frac{1}{\text{const}}\, \omega_\Phi^2(f; \delta)_p \leqslant K(f; \delta^2; L^p, W_\Phi^{p,2}) \leqslant \text{const}\, \omega_\Phi^2(f; \delta)_p.$$

We shall prove a more general version of this lemma along with an application to the Bernstein polynomials on simplices in a subsequent paper [4].

The following theorem is an immediate consequence of Lemma 6 and our results presented above.

THEOREM 4. *Let $f \in L^p(S)$, $1 \leqslant p < +\infty$. Then*

$$\forall n \in \mathbb{N}_o, \qquad \| V_n f - f \|_p \leqslant \text{const} \left\{ \frac{\| f \|_p}{n+1} + \omega_\Phi^2(f; 1/\sqrt{n+1})_p \right\}.$$

Furthermore, for $f \in L^2(S)$,

$$\omega_\Phi^2(f; 1/\sqrt{n+1})_2 \leqslant \text{const} \max_{n \leqslant k} \{ \| V_k f - f \|_2 \}.$$

We conjecture that the inverse part of the theorem remains true for all p, $1 < p < +\infty$.

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REFERENCES

1. P. APPELL AND J. KAMPÉ DE FÉRIET, "Fonctions hypergéométriques et hypersphériques - Polynômes d'Hermite," Gauthier-Villars, Paris, 1926.
2. H. BERENS AND Y. XU, On Bernstein–Durrmeyer polynomials with Jacobi-weights, *in* "Approximation Theory and Functional Analysis" (C. K. Chui, Ed.), pp. 25–46, Academic Press, Boston, 1991.
3. H. BERENS AND Y. XU, On Bernstein–Durrmeyer polynomials with Jacobi-weights: The cases $p = 1$ and $p = \infty$, *in* "Approximation, Interpolation, and Summation" (S. Baron and D. Leviatan, Eds.), Israel Mathematical Conference Proceedings, Vol. 4, pp. 51–62, Weizmann Science Press of Israel, Jerusalem, 1991.
4. H. BERENS AND Y. XU, K-Moduli, moduli of smoothness, and Bernstein polynomials on a simplex, *Indag. Mathem.* N.S. **2**(3) (1991), in print.
5. M. M. DERRIENNIC, Sur l'approximation de fonctions intégrables sur [0, 1] par des polynômes de Bernstein modifiés, *J. Approx. Theory* **31** (1981), 325–343.

6. M. M. DERRIENNIC, On multivariate approximation by Bernstein-type polynomials, *J. Approx. Theory* **45** (1985), 155–166.
7. Z. DITZIAN AND K. IVANOV, Bernstein-type operators and their derivatives, *J. Approx. Theory* **56** (1989), 72–90.
8. Z. DITZIAN AND V. TOTIK, "Moduli of Smoothness," SSCM, Vol. 9, Springer-Verlag, Berlin, 1987.